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Symmetric functions and the KP and BKP hierarchies

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Abstract. We study the KP hierarchy through its relationship with S-functions. Using results from the classical theory of symmetric functions, the Plücker equations for the hierarchy are derived from the tau function bilinear identity and are given in terms of composite S-functions. Their connection to the Hirota bilinear form of the hierarchy is clarified. A novel combinatorial proof is given of the fact that Schur polynomials solve the KP hierarchy. We show how the analysis can be carried through for the BKP hierarchy in a completely parallel fashion, with the S-functions replaced by Schur Q-functions.

1. Introduction

Vertex operators have come to play an important role in various areas of mathematics and physics [1], including conformal field theory, integrable hierarchies and the representation theory of Kac–Moody algebras. In [2, 3] a realization of untwisted vertex operators as operations on Schur functions (S-functions) was presented. It was shown how classical results from the theory of symmetric functions can be effectively used to calculate quantities like matrix elements and traces of products of vertex operators, with results succintly expressed in terms of composite Schur and supersymmetric Schur functions. In this paper, we explore the implications of this realization of vertex operators for integrable systems of the KP (Kadomtsev–Petviashvili) type [4].

Schur polynomials are known to solve the KP hierarchy [5]. It thus makes sense to study the KP hierarchy from a point of view which stresses this property. It will be seen that our novel approach has its benefits, not the least of which is a straightforward generalization to the BKP hierarchy [6] with the role of S-functions played by Schur Q-functions, Schur Q-polynomials being solutions to the BKP hierarchy [7, 8]. Our treatment starts from the tau function bilinear identity (or group-orbit equation) which traditionally leads to the formulation of the hierarchy in Hirota bilinear form. However we take a different route, through a change of variables which effectively implements the Schur function realization of the vertex operators which define the infinite dimensional algebra underlying the hierarchy. Through a sequence of simple manipulations of the S-functions, we convert the bilinear identity to a set of Plücker equations (which are central to the infinite dimensional Grassmannian approach to the KP hierarchy [5]). Our version of the Plücker equations is formulated in terms of composite S-functions and turns out to be slightly different to the Sato version quoted in [4]. The fact that Schur polynomials solve the KP hierarchy is related to a property of composite S-functions (of a single variable), which we prove in appendix A. We show how the Plücker equations can be converted into bilinear PDEs for the tau function and how these PDEs are related to the Hirota bilinear form of the hierarchy. We also clarify (in appendix B) the relationship between Hirota derivatives and

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supersymmetric polynomials. Finally we show how to obtain the well known N-soliton solutions in our language. This is all done in section 2. The completely parallel treatment of the BKP hierarchy is done in section 3.

2. The KP hierarchy and S-functions

2.1. Review of the KP hierarchy in Hirota bilinear form

The KP hierarchy [5] is an important integrable system, whose reductions include the famous KdV and Boussinesq hierarchies, and is usually formulated in the Lax form

$$\frac{\partial L}{\partial x_n} = [L_+^n, L] \tag{2.1}$$

where $L = \partial + \sum_{i=1}^{\infty} u_i \partial^{-i}$ ($\partial \equiv \partial/\partial x_1$) and P_+ denotes the differential operator part of a pseudodifferential operator P. Through a dressing transformation $L = W\partial W^{-1}$, with $W = 1 + \sum_{i=1}^{\infty} w_i \partial^{-i}$ one arrives at the equivalent Sato form of the hierarchy

$$\frac{\partial W}{\partial x_n} = B_n W - W\partial^n$$

where $B_n = (L^n)_+$. The system (2.1) is also equivalent to the linear system

$$Lw = zw \qquad \frac{\partial w}{\partial x_n} = B_n w \qquad \frac{\partial z}{\partial x_n} = 0\,.$$

Up to a function of the spectral parameter z, the wavefunction $w(x,z)$ is given by $w(x,z) = W\exp(\sum x_k z^k)$ and yet another equivalent form of the hierarchy is the 'bilinear identity'

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} w(x,z) w^*(y,z) = 0 \tag{2.2}$$

where $w^*(x,z) = W^*\exp(\sum x_k z^k)$ is the adjoint wavefunction with W^* being the formal adjoint of W. Finally, given a wavefunction $w(x,z)$, a tau function can be obtained as

$$w(x,z) = \frac{\exp\left(\sum_k x_k z^k\right)}{\tau(x)} \exp\left(-\sum_k \frac{z^{-k}}{k}\frac{\partial}{\partial x_k}\right)\tau(x)$$

resulting in the tau function form of the hierarchy

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \exp\left(\sum_{k=1}^{\infty} z^k (x_k - y_k)\right) \exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k}\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial y_k}\right)\right)\tau(x)\tau(y) = 0\,. \tag{2.3}$$

Alternatively, equation (2.3) can be interpreted [9] as the condition that τ must satisfy if it lies in the $GL(\infty)$-orbit of the vacuum vector in a vertex representation of a_∞ on $\mathbb{C}[x_1, x_2, \ldots]$. The exponential factors in (2.3) come from bosonization of the fermions, normal-ordered bilinears of which generate a_∞. For a review, see [4] or [10]. This group orbit interpretation of the KP hierarchy can be carried over to many other integrable systems. In fact, one can start with a given vertex representation of an infinite dimensional algebra and construct a corresponding hierarchy [11]. Integrability of many of these hierarchies has now been established by virtue of the existence of Lax representations [12].

The change of variables $x \to x - y$, $y \to x + y$ leads directly to the Hirota bilinear form of the hierarchy (or generating functions for the Hirota equations)†

$$\sum_m S_m(-2y) S_{m+1}(\tilde{D}) \exp\left(\sum y_k D_k\right)(\tau \cdot \tau) = 0 \tag{2.4}$$

where S_m are elementary Schur polynomials whose generating function is

$$\exp\left(\sum_{k=1}^{\infty} x_k z^k\right) = \sum_{k=0}^{\infty} S_k(x) z^k$$

$\tilde{D} = (\frac{D_1}{1}, \frac{D_2}{2}, \frac{D_3}{3}, \dots)$ and D_k is a Hirota derivative defined by

$$P(D_k)(\tau \cdot \tau) = P\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial y_k}\right) \tau(x)\tau(y)\bigg|_{y=x}$$

for any polynomial P. Our analysis of the KP hierarchy begins with (2.3).

2.2. *Review of symmetric functions*

We recall here various properties of symmetric functions [14]. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of $|\lambda| \equiv \sum_{i=1}^{n} \lambda_i$ into $l(\lambda) \equiv n$ parts with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, the S-function $s_\lambda(x)$ of an infinite number of indeterminates $x = (x_1, x_2, \dots)$ is defined as

$$s_\lambda(x) = \det(h_{\lambda_i - i + j}(x))$$

with $h_m(x)$ being complete symmetric functions whose generating function is

$$\sum_{m=0}^{\infty} h_m(x) t^m = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} .$$

The set of S-functions $s_\lambda(x)$ where λ runs over all partitions forms an integral basis for the ring Λ of symmetric functions of x with integer coefficients. An inner product can be defined on Λ such that the S-functions are orthonormal: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Another important orthogonal basis is provided by the power sum symmetric functions $p_\lambda \equiv p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$ where $p_m(x)$ has the generating function

$$\sum_{m=1}^{\infty} p_m(x) t^{m-1} = \frac{\mathrm{d}}{\mathrm{d}t} \log \prod_{i=1}^{\infty} (1 - x_i t)^{-1}. \tag{2.5}$$

With respect to the same inner product, we have $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$, where $z_\lambda = \prod_{i \geqslant 1} i^{m_i} m_i!$ with $m_i = m_i(\lambda)$ being the number of parts of λ equal to i. Let $D(f)$ for any symmetric function f denote the adjoint of multiplication by f. Then there is the result [14]

$$D(p_n(x)) = n \frac{\partial}{\partial p_n(x)}. \tag{2.6}$$

The operations in Λ of multiplication by p_m and its adjoint therefore provides a realization of the Heisenberg algebra. This simple observation together with the observation that certain vertex operators in this realization are 'simplest' in the S-function basis was the starting

† Closed form expressions for the Hirota polynomials for the KP and other hierarchies can be obtained using symmetric function techniques [13].

point of the works [2, 3]. The relation between the S-functions and the power sums is most easily seen via the following S-function series:

$$J(x;y) \equiv \sum_{\lambda} s_\lambda(x)s_\lambda(y) = \prod_{i,j=1}^{\infty}(1-x_iy_j)^{-1} \tag{2.7}$$

which together with (2.5) implies

$$\sum_{\lambda} s_\lambda(x)s_\lambda(y) = \exp\left(\sum_{m=1}^{\infty}\tfrac{1}{m}p_m(x)p_m(y)\right). \tag{2.8}$$

Further on, we will make use of another S-function series

$$I(x;y) \equiv \sum_{\lambda}(-1)^{|\lambda|}s_\lambda(x)s_{\lambda'}(y) = \prod_{i,j=1}^{\infty}(1-x_iy_j) \tag{2.9}$$

with λ' denoting the transpose of λ.

We summarize here various S-function properties which are useful later on. Firstly, S-function multiplication is given by

$$s_\mu(x)s_\nu(x) = \sum_{\lambda} c^{\lambda}_{\mu\nu}s_\lambda(x) \tag{2.10}$$

with the coefficients $c^{\lambda}_{\mu\nu}$ determined by the Littlewood–Richardson rule. Skew S-functions are defined as

$$s_{\lambda/\mu}(x) = \sum_{\nu} c^{\lambda}_{\mu\nu}s_\nu(x) \tag{2.11}$$

or, equivalently, $s_{\lambda/\mu}(x) = D(s_\mu(x))s_\lambda(x)$. In particular $s_{\lambda/0}(x) = s_\lambda(x)$ and $s_{\lambda/\mu}(x)$ is non-zero only if the Young diagram corresponding to the partition μ lies within that corresponding to λ. If the compound argument (x, y) denotes the set of variables $(x_1, x_2, \ldots, y_1, y_2, \ldots)$, then we have the relation

$$s_\lambda(x,y) = \sum_{\mu} s_{\lambda/\mu}(x)s_\mu(y)\,.$$

In working with vertex operators in the Schur function realization, supersymmetric S-functions and composite S-functions naturally appear. Supersymmetric S-functions in the variables x and y are defined by

$$s_\lambda(x/y) = \sum_{\mu}(-1)^{|\mu|}s_{\lambda/\mu}(x)s_{\mu'}(y) \tag{2.12}$$

and obey the same rule (2.10) with the argument x replaced by the supersymmetric argument x/y. Composite S-functions are defined by

$$s_{\bar{\nu};\mu}(x) = \sum_{\xi}(-1)^{|\xi|}s_{\nu/\xi'}(\tfrac{1}{x})s_{\mu/\xi}(x). \tag{2.13}$$

Both these S-functions play important roles in the Young diagrammatic approach to Lie (super) algebra representation theory. The characters for representations of $sl(n)$, whose highest weight vectors are labelled by partitions, are precisely given by S-functions. Composite S-functions provide an alternative set, and in the case of $gl(n)$ are associated with more general representations. For a review, we refer the reader to [15].

2.3. From the bilinear identity to Plücker equations

It is a key result of the KP theory that Schur polynomials $S_\lambda(x)$, defined by

$$S_\lambda(x) = \det\left(S_{\lambda_i - i + j}(x)\right)$$

solve the KP hierarchy [5]. These Schur polynomials are related to the S-functions by $S_\lambda(x) = s_\lambda(u)$ where $x_k = \frac{1}{k} p_k(u)$ with $u = (u_1, u_2, \ldots)$ etc. Note that the 'S' for Schur polynomials is capitalized as opposed to the 's' for S-functions. The change of variables

$$x_k = \tfrac{1}{k} p_k(u) \qquad y_k = \tfrac{1}{k} p_k(v) \tag{2.14}$$

is hence a very natural one to make on the bilinear identity (2.3) itself. Due to (2.6) this change of variables is accompanied by

$$\frac{\partial}{\partial x_k} = D(p_k(u)) \qquad \frac{\partial}{\partial y_k} = D(p_k(v)) \,.$$

Here we have used D to signify adjoint with respect to the inner products on two different spaces but this should cause no confusion. Together with the device of 'promoting' z to a multivariable $z = (z_1, z_2, \ldots)$ where eventually we will set $z_1 = z$, $0 = z_2 = z_3 = \cdots$, the bilinear identity (2.3) becomes

$$0 = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \exp\left(\sum_{m=1}^{\infty} \tfrac{1}{m} p_m(z) p_m(u)\right) \exp\left(-\sum_{m=1}^{\infty} \tfrac{1}{m} p_m(\tfrac{1}{z}) D(p_m(u))\right)$$
$$\times \exp\left(-\sum_{m=1}^{\infty} \tfrac{1}{m} p_m(z) p_m(v)\right) \exp\left(\sum_{m=1}^{\infty} \tfrac{1}{m} p_m(\tfrac{1}{z}) D(p_m(v))\right) \tau(u)\tau(v) \,.$$

By an abuse of notation, τ is now to be considered as a function of u - a symmetric function, in fact. Using (2.8) and the corresponding relation which follows from (2.9), we obtain

$$0 = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left(\sum_\lambda s_\lambda(u) s_\lambda(z)\right) \left(\sum_\rho (-1)^{|\rho|} s_{\rho'}(\tfrac{1}{z}) D(s_\rho(u))\right) \left(\sum_\mu (-1)^{|\mu|} s_\mu(v) s_{\mu'}(z)\right)$$
$$\times \left(\sum_\nu s_\nu(\tfrac{1}{z}) D(s_\nu(v))\right) \tau(u)\tau(v) \,. \tag{2.15}$$

This can be written in a more compact form in terms of supersymmetric S-functions

$$0 = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left(\sum_\upsilon s_\upsilon(u/v) s_\upsilon(z)\right) \left(\sum_\xi D(s_\xi(u/v) s_\xi(\tfrac{1}{z})\right) \tau(u)\tau(v) \,.$$

Since $\tau(u)$ is a symmetric function, it can be written as a linear combination of S-functions

$$\tau(u) = \sum_\lambda a^\lambda s_\lambda(u). \tag{2.16}$$

In terms of the original KP variables x_k this is equivalent to $\tau(x) = \sum_\lambda a^\lambda S_\lambda(x)$. We now proceed to reduce the condition (2.15) for τ to be a solution of the KP hierarchy to algebraic equations for a^λ. The method is essentially that used in [2, 3] to calculate vertex operator matrix elements in the S-function basis. By the definition of skew S-functions we rewrite (2.15) with τ given by (2.16) as

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \sum s_\lambda(u) s_\lambda(z)(-1)^{|\rho|} s_{\rho'}(\tfrac{1}{z}) a^\eta s_{\eta/\rho}(u)(-1)^{|\mu|} s_\mu(v) s_{\mu'}(z) s_\nu(\tfrac{1}{z}) a^\xi s_{\xi/\nu}(v) = 0\,.$$

By using (2.11) to expand out the skew S-functions of u and v, then (2.10) to multiply together the S-functions of u and v, and finally (2.11) again to re-sum in favour of skew S-functions of z and z^{-1} we obtain

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \sum s_\beta(u) s_{\beta/\sigma}(z) s_{\eta'/\sigma'}(-\tfrac{1}{z}) a^\eta s_\alpha(v) s_{\alpha'/\chi'}(-z) s_{\xi/\chi}(\tfrac{1}{z}) a^\xi = 0\,.$$

The terms in z and z^{-1} can be recognized as precisely the combinations used to define the composite S-functions (2.13). This, together with the projecting out of the S-functions in u and v leads us to the Plücker equations

$$\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \sum_{\eta,\xi} (-1)^{|\eta|+|\alpha|} a^\eta a^\xi s_{\bar{\eta}';\beta}(z) s_{\bar{\xi};\alpha'}(z) = 0 \tag{2.17}$$

which must be satisfied for all α and β if (2.16) is to solve the KP hierarchy.

We recall that z is such that $z_2 = z_3 = \cdots = 0$. S-functions of a single variable are simple: $s_\lambda(z)$ is non-zero only if λ is a one-part partition and $s_{(m)}(z) = h_m(z)$, a complete symmetric function. Similarly, skew S-functions of a single variable are easy to calculate [14]: $s_{\lambda/\mu}(z)$ is non-zero only if the skew diagram $\lambda - \mu$ is a 'horizontal strip'—i.e. if $\lambda_i' - \mu_i' \leqslant 1$ for all i—in which case $s_{\lambda/\mu}(z) = z^{|\lambda|-|\mu|}$. The composite S-functions appearing in (2.17) can thus be calculated explicitly. The integral in (2.17) also poses no problems. By explicit calculation of the composite S-functions, the non-trivial Plücker equations for all α and β with $|\alpha| + |\beta| \leqslant 5$ are given by†

$$\begin{aligned}
&a^{(0)}a^{(2,2)} + a^{(1,1)}a^{(2)} - a^{(2,1)}a^{(1)} = 0\\
&a^{(0)}a^{(3,2)} + a^{(1,1)}a^{(3)} - a^{(3,1)}a^{(1)} = 0\\
&a^{(0)}a^{(2,2,1)} + a^{(1,1,1)}a^{(2)} - a^{(2,1,1)}a^{(1)} = 0\\
&a^{(0)}a^{(4,2)} + a^{(1,1)}a^{(4)} - a^{(4,1)}a^{(1)} = 0\\
&a^{(0)}a^{(3,3)} + a^{(2,1)}a^{(3)} - a^{(3,1)}a^{(2)} = 0\\
&a^{(0)}a^{(3,2,1)} + a^{(1,1,1)}a^{(3)} - a^{(3,1,1)}a^{(1)} = 0\\
&a^{(0)}a^{(2,2,1,1)} + a^{(1,1,1,1)}a^{(2)} - a^{(2,1,1,1)}a^{(1)} = 0\\
&a^{(0)}a^{(2,2,2)} + a^{(1,1,1)}a^{(2,1)} - a^{(2,1,1)}a^{(1,1)} = 0\,.
\end{aligned} \tag{2.18}$$

Alternatively, $s_{\bar{\nu};\mu}(z)$ is the character of the $gl(1)$ representation labelled by the composite Young diagram $\{\bar{\nu};\mu\}$ and there exists sophisticated 'modification rules' to calculate it. These calculations have in fact been automated in the program SCHUR [17]. Our claim that these are the Plücker equations for the KP hierarchy is based on a comparison with the first few Plücker equations of Sato, quoted in [4]. There the general Plücker equation has strictly three terms each with coefficient ± 1. As shown in appendix A, the Plücker equations in our form (2.17) can have more than three terms (for α, β of high enough weight). Presumably the two forms for the Plücker equations correspond to choosing different bases.

Note that the lack of quadratic terms $(a^\lambda)^2$ in the Plücker equations is due to the fact that Schur polynomials solve the KP hierarchy [5]. In other words, for any given λ, $a^\alpha = \delta_{\alpha\lambda}$ is a solution to (2.17). In appendix A we provide a proof of this fact.

† These were obtained with a program written in Mathematica [16].

One can in fact obtain bilinear equations in $\tau(x)$ of similar form to the Plücker equations. For suppose τ is a function of the compound argument (u, w). Then we have an expansion $\tau(u, w) = \sum_\alpha a^\alpha s_\alpha(u, w)$, which on using (2.13) becomes

$$\tau(u, w) = \sum_{\alpha,\beta} a^\alpha s_{\alpha/\beta}(u) s_\beta(w) .$$

Now, one can carry through the analysis leading to the Plücker equations as before but now with $\tau(u, w)$ thought of as a function of w, with the result that $a^\beta(u) \equiv \sum_\beta a^\alpha s_{\alpha/\beta}(u)$ satisfies the same set of equations (2.17). But there is the result

$$a^\beta(u) = S_\beta(\tilde{\partial}_x)\tau(x) \tag{2.19}$$

with $\tilde{\partial} = (\frac{1}{1}\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \ldots)$, which means that every Plücker equation has a bilinear PDE equivalent. To prove (2.20), we note that since $a^\beta(u) = D(s_\beta(u))\tau(u)$, it has a generating function

$$\begin{aligned} \sum_\beta a^\beta(u) s_\beta(t) &= D\left(\sum_\beta s_\beta(t) s_\beta(u)\right)\tau(u) \\ &= \exp\left(\sum_{m=1}^{\infty} p_m(t)\frac{\partial}{\partial p_m(u)}\right)\tau(u) \\ &= \sum_\beta s_\beta(t) S_\beta(y)\tau(u) \end{aligned}$$

for a fictitious set of variables $y_m = \frac{1}{m}\partial/\partial p_m(u)$. On making the transformation (2.14) back to the usual KP variables, (2.20) follows. Alternatively, (2.20) can be derived by making use of the Frobenius formula

$$s_\rho(t) = \sum_\lambda z_\lambda^{-1}\chi_\lambda^\rho p_\lambda(t) \tag{2.20}$$

where χ_λ^ρ is a symmetric group character, and the relation (2.6).

Corresponding to the first equation of (2.19) we have

$$\begin{aligned} &\left(S_{(0)}(\tilde{\partial})\tau(x)\right)\left(S_{(2,2)}(\tilde{\partial})\tau(x)\right) + \left(S_{(2)}(\tilde{\partial})\tau(x)\right)\left(S_{(1,1)}(\tilde{\partial})\tau(x)\right) \\ &\qquad - \left(S_{(2,1)}(\tilde{\partial})\tau(x)\right)\left(S_{(1)}(\tilde{\partial})\tau(x)\right) = 0 . \end{aligned}$$

When the Schur polynomials are evaluated it can be seen that this equation is the same as

$$(4D_1D_3 - D_1^4 - 3D_2^2)\tau \cdot \tau = 0$$

the Hirota bilinear form of the KP equation. The bilinear equivalent of the next two equations of (2.19) cannot be written in Hirota bilinear form separately since each contains the term $\tau(\partial_{x_1}^5\tau)$ whereas the Hirota bilinear operator D_1^5 is trivial as are all odd $P(D)$. However, their difference yields the second equation of the KP hierarchy

$$(D_1^3D_2 - 3D_1D_4 + 2D_2D_3)\tau \cdot \tau = 0 .$$

As shown in appendix B, a Hirota term of the form $D_\lambda(\tau\cdot\tau)$, where $D_\lambda = D_{\lambda_1}D_{\lambda_2}\cdots D_{\lambda_n}$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, can be written as a bilinear PDE

$$\sum_\rho \chi_\lambda^\rho\, S_\rho\left(\tilde{\partial}_x/\tilde{\partial}_y\right)\tau(x)\tau(y)\Big|_{y=x}$$

in which the role of supersymmetric S-functions is manifest. The Hirota bilinear equations arising from the generating function (2.4) are then equivalent to linear combinations of such symmetrized bilinear PDEs. Since both (2.4) and the Plücker equations (2.17) are consequences of the bilinear identity (2.3), we expect that linear combinations of the Plücker equations can be found to yield the symmetrized bilinear PDEs which correspond to the Hirota bilinear equations.

2.4. Soliton and other solutions

The vertex operator [9] $\Gamma(a,b) = \exp(\sum_j (a^j - b^j)x_j)\exp(-\sum_j((a^{-j} - b^{-j})/j)(\partial/\partial x_j))$ defines the vertex representation of a_∞ on $\mathbb{C}[x_1, x_2, \ldots]$ which underlies the KP hierarchy. Exponentiating elements of a_∞ we get elements of $GL(\infty)$ and, since the KP hierarchy is the $GL(\infty)$-orbit, their action on solutions of the KP hierarchy yield new solutions. Now, we already know that any Schur polynomial is a solution. The action on them by products of exponentials of $\Gamma(a,b)$ is particularly interesting. Such operators can be written as

$$\prod_i \exp(c_i\Gamma(a_i,b_i)) = \prod_i (1 + c_i\Gamma(a_i,b_i))$$

since $\Gamma(a,b)^2 = 0$.

By the transformation $x_k \to \frac{1}{k}p_k(u)$ and promoting a and b to $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ we rewrite $\Gamma(a,b)$ as

$$\begin{aligned}\Gamma(a,b) &= \exp\left(\sum_m \tfrac{1}{m}p_m(a/b)p_m(u)\right)\exp\left(\sum_m \tfrac{1}{m}p_m(\tfrac{1}{a}/\tfrac{1}{b})D(p_m(u))\right)\\ &= \sum_{\lambda,\mu}(-1)^{|\mu|}s_\lambda(a/b)s_\lambda(u)s_{\mu'}(\tfrac{1}{a}/\tfrac{1}{b})D(s_\mu(u))\end{aligned} \tag{2.21}$$

and its action on an S-function is given by

$$\Gamma(a,b)s_\alpha(u) = (-1)^{|\alpha|}\sum_\beta s_{\bar{\alpha}';\beta}(a/b)s_\beta(u)\ .$$

More generally, we have

$$\prod_{i=1}^n \Gamma(a_i,b_i)s_\alpha(u)$$

$$= (-1)^{|\alpha|}\prod_{1\leqslant i<j\leqslant n} I\left(a_i, b_i;\ \tfrac{1}{a_j}\tfrac{1}{b_j}\right)\sum_\beta s_{\bar{\alpha}';\beta}(a_1,\ldots,a_n/b_1,\ldots,b_n)s_\beta(u) \tag{2.22}$$

where

$$I(x/w;\ y/z) = I(x;\ y)I(w;\ z)J(x;\ z)J(w;\ y)$$

and $I(x;y)$, etc, are defined in (2.9). Note now that

$$\prod_{i=1}^N (1 + c_i\Gamma(a_i,b_i)) = \sum_{n=0}^N e_n(c_1\Gamma(a_1,b_1), c_2\Gamma(a_2,b_2), \ldots)$$

where $e_n(x)$ is the elementary symmetric function $e_n(x) = \sum_{1\leqslant i_1<\cdots<i_n\leqslant N} x_{i_1}\cdots x_{i_n}$. Hence the action of a product of exponentials of $\Gamma(a_i, b_i)$ on an S-function is

$$\prod_{i=1}^{N}(1+c_i\Gamma(a_i,b_i))\,s_\alpha(u) = (-1)^{|\alpha|}\sum_{n=0}^{N}\sum_{1\leqslant i_1<\cdots<i_n\leqslant N} c_{i_1}\cdots c_{i_n}$$
$$\times \prod_{1\leqslant\eta<\zeta\leqslant n} I\left(a_{i_\eta}/b_{i_\eta};\, \tfrac{1}{a_{i_\zeta}}/\tfrac{1}{b_{i_\zeta}}\right)\sum_{\gamma} s_{\tilde{\alpha}';\gamma}(a_{i_1},\ldots,a_{i_n}/b_{i_1},\ldots,b_{i_n})s_\gamma(u) \tag{2.23}$$

which solves the KP hierarchy once the transformation back to the KP variables x_k is made. The special case $\alpha = 0$ corresponds to the famous N-soliton solution, for then we have

$$\prod_{i=1}^{N}(1+a_i\Gamma(a_i,b_i))\cdot 1 = \sum_{n=0}^{N}\sum_{1\leqslant i_1<\cdots<i_n\leqslant N} c_{i_1}\cdots c_{i_n}$$
$$\times \prod_{1\leqslant\eta<\zeta\leqslant n} I\left(a_{i_\eta}/b_{i_\eta};\, \tfrac{1}{a_{i_\zeta}}/\tfrac{1}{b_{i_\zeta}}\right)\exp\left(\sum_{m=1}^{\infty}\sum_{\alpha=1}^{n}(a_{i_\alpha}^m - b_{i_\alpha}^m)x_m\right). \tag{2.24}$$

3. The BKP hierarchy and Q-functions

3.1. Preliminaries

The BKP is a KP-like hierarchy where instead of a_∞, the underlying infinite dimensional algebra is b_∞. It admits a Lax representation of the type (2.1) with an extra condition $L = -\partial^{-1}L^*\partial$ on the Lax operator L, at the expense of freezing out the variables $x_2, x_4, \ldots$ The wavefunction in this case satisfies the bilinear identity [6]

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i}z} w(x,z)w(y,-z) = 1\,.$$

One can consistently set

$$w(x,z) = \frac{\exp\left(\sum_{k\text{ odd}} x_k z^k\right)}{\tau(x)}\exp\left(-2\sum_{k\text{ odd}}\frac{z^{-k}}{k}\frac{\partial}{\partial x_k}\right)\tau(x)$$

resulting in the tau function bilinear identity

$$\tau(x)\tau(y) = \oint\frac{\mathrm{d}z}{2\pi\mathrm{i}z}\exp\left(\sum_{l\text{ odd}} z^l(x_l-y_l)\right)\exp\left(-2\sum_{l\text{ odd}}\frac{z^{-l}}{l}\left(\frac{\partial}{\partial x_l}-\frac{\partial}{\partial y_l}\right)\right)\tau(x)\tau(y) \tag{3.1}$$

which can be interpreted as the $O(\infty)$-orbit equation for the vacuum vector in a vertex representation of b_∞ on $\mathbb{C}[x_1, x_3, \ldots]$.

For the BKP hierarchy, the relevant set of symmetric functions turns out to be the Schur Q-functions, first introduced in the context of projective representations of symmetric groups. They form a basis for the ring Γ of symmetric functions spanned by the power sums $p_m(x)$ for odd m. One definition of Q-functions is the following [14]:

$$Q_\lambda(x) = \prod_{j<k}\frac{1-R_{jk}}{1+R_{jk}}q_\lambda(x) \tag{3.2}$$

where $q_\lambda(x) = q_{\lambda_1}(x)q_{\lambda_2}(x)\ldots q_{\lambda_n}(x)$ and the generating function for $q_m(x)$ is

$$\sum_{m=0}^{\infty} q_m(x)y^m = \prod_{i=1}^{\infty} \frac{(1+x_i y)}{(1-x_i y)} \,.$$

The Young operators R_{jk} act on l-tuples, adding 1 to the jth component and subtracting 1 from the kth component. Its action on q_λ is $R_{jk}q_\lambda = q_{R_{jk}(\lambda)}$. So, for instance

$$\begin{aligned} Q_{(m,n)} &= \frac{1-R_{12}}{1+R_{12}} q_{(m,n)} \\ &= (1 - 2R_{12} + 2R_{12}^2 - 2R_{12}^3 + \cdots)q_{(m,n)} \\ &= q_m q_n - 2q_{m+1}q_{n-1} + \cdots + (-1)^n 2q_{m+n} \,. \end{aligned}$$

Note that the S-functions can be similarly defined:

$$s_\lambda(x) = \prod_{i<j}(1 - R_{ij})h_\lambda \,.$$

There are two other equivalent definitions of Schur Q-functions. See for instance [18]. The definition (3.2) is the one which generalizes to Hall–Littlewood symmetric functions.

The Q-functions are non-zero only if λ belongs to the set DP of partitions with all parts distinct. By definition, $Q_\lambda(x)$ is homogeneous in x_i with degree $|\lambda|$. The Q-functions satisfy a relation analogous to (2.7), namely

$$\sum_{\lambda\in DP} \frac{1}{b_\lambda} Q_\lambda(x)Q_\lambda(y) = \prod_{i,j=1}^{\infty} \frac{(1+x_i y_j)}{(1-x_i y_j)} \tag{3.3}$$

with $b_\lambda = 2^{l(\lambda)}$. Using (2.5) it can be shown that the relation

$$\prod_{i,j=1}^{\infty} \frac{(1+x_i y_j)}{(1-x_i y_j)} = \exp\left(2\sum_{m \text{ odd}} \frac{1}{m} p_m(x)p_m(y)\right) \tag{3.4}$$

also holds. This provides the desired link between Q-functions and odd power sums

$$\sum_{\lambda\in DP} \frac{1}{b_\lambda} Q_\lambda(x)Q_\lambda(y) = \exp\left(2\sum_{m \text{ odd}} \frac{1}{m} p_m(x)p_m(y)\right). \tag{3.5}$$

There exists an inner product with respect to which the Q-functions are orthogonal:

$$\langle Q_\lambda(x), Q_\mu(x)\rangle = b_\lambda \delta_{\lambda\mu} \,. \tag{3.6}$$

This inner product in fact differs from the previous inner product for the S-functions by a factor of 2. In particular, we now have

$$D(p_m(x)) = \frac{m}{2}\frac{\partial}{\partial p_m(x)} \,. \tag{3.7}$$

Q-function multiplication is given by $Q_\lambda(x)Q_\mu(x) = \sum_{\nu\in DP} F^\nu_{\lambda\mu} Q_\nu(x)$, whereas skew Q-functions are defined by $Q_{\lambda/\mu}(x) = \sum_{\nu\in DP} f^\lambda_{\mu\nu} Q_\nu(x)$ with $f^\lambda_{\mu\nu} = b_\lambda b_\mu^{-1} b_\nu^{-1} F^\lambda_{\mu\nu}$. Alternatively, $Q_{\lambda/\mu}(x)$ is obtained as $b_\mu^{-1} D(Q_\mu(x))Q_\lambda(x)$.

Schur Q-polynomials $Q_\lambda(x)$ are related to the Schur Q-functions in the same way that Schur polynomials are related to the Schur S-functions. Given $x_m = \frac{2}{m}p_m(u)$, define† $\overline{Q}_\lambda(x)$ to be such that $\overline{Q}_\lambda(x) = Q_\lambda(u)$. The right-hand side is calculated using the definition (3.2) with $q_m(u)$ related to the elementary Schur polynomials by $q_m(u) = S_m(\bar{x})$, since

$$\sum_{m=0}^{\infty} t^m q_m(u) = \exp\left(2\sum_{m\ \mathrm{odd}} \frac{t^m}{m} p_m(u)\right) = \sum_{m=0}^{\infty} S_m(\bar{x}) t^m \tag{3.8}$$

where $\bar{x}$ denotes $\bar{x} = (x_1, 0, x_3, 0, x_5, \ldots)$. The first few Schur Q-polynomials are: $Q_{(0)}(x) = 1$, $Q_{(1)}(x) = x_1$, $Q_{(2)}(x) = x_1^2/2$, $Q_{(3)}(x) = x_1^3/6 + x_3$, $Q_{(2,1)}(x) = x_1^3/6 - 2x_3$, $Q_{(3,1)}(x) = x_1^4/12 - x_1x_3$, $Q_{(3,2)}(x) = x_1^5/60 - x_1^2x_3/2 + 2x_5$, and $Q_{(3,2,1)}(x) = x_1^6/360 - x_1^3x_3/6 - 2x_3^2 + 2x_1x_5$.

3.2. Plücker equations

As for the KP hierarchy in subsection 2.3, we perform a change of variables on the bilinear identity (3.1)—this time being

$$x_k = \tfrac{2}{k}\, p_k(u) \qquad y_k = \tfrac{2}{k}\, p_k(v) \tag{3.9}$$

(for odd k) and, correspondingly

$$\frac{\partial}{\partial x_k} = D(p_k(u)) \qquad \frac{\partial}{\partial y_k} = D(p_k(v))\,.$$

Promoting z to a multivariable $z = (z_1, z_2, \ldots)$ as before, we obtain

$$\tau(u)\tau(v) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \exp\left(2\sum_{k\ \mathrm{odd}} \tfrac{1}{k} p_k(z) p_k(u)\right) \exp\left(-2\sum_{k\ \mathrm{odd}} \tfrac{1}{k} p_k(\tfrac{1}{z}) D(p_k(u))\right)$$

$$\times \exp\left(-2\sum_{k\ \mathrm{odd}} \tfrac{1}{k} p_k(z) p_k(v)\right) \exp\left(2\sum_{k\ \mathrm{odd}} \tfrac{1}{k} p_k(\tfrac{1}{z}) D(p_k(v))\right) \tau(u)\tau(v)\,.$$

Making use of the identity (3.5) and a corresponding identity

$$\exp\left(-2\sum_{k\ \mathrm{odd}} \tfrac{1}{k} p_k(z) p_k(u)\right) = \sum_{\lambda \in DP} \frac{1}{b_\lambda} (-1)^{|\lambda|} Q_\lambda(z) Q_\lambda(u)$$

obtained by making the transformation $y \to -y$ in (3.5), we obtain the bilinear identity in the form

$$\tau(u)\tau(v) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \left(\sum_\lambda \tfrac{1}{b_\lambda} Q_\lambda(z) Q_\lambda(u)\right) \left(\sum_\mu \tfrac{1}{b_\mu} (-1)^{|\mu|} Q_\mu(\tfrac{1}{z}) D(Q_\mu(u))\right)$$

$$\times \left(\sum_\nu \tfrac{1}{b_\nu} (-1)^{|\nu|} Q_\nu(z) Q_\nu(v)\right) \left(\sum_\eta \tfrac{1}{b_\eta} Q_\eta(\tfrac{1}{z}) D(Q_\eta(v))\right) \tau(u)\tau(v) \tag{3.10}$$

Now if we *define* supersymmetric Q-functions to be

$$Q_\lambda(x/y) = \sum_{\nu \in DP} (-1)^{|\nu|} Q_{\lambda/\nu}(x) Q_\nu(y) \tag{3.11}$$

† Actually, we will drop the bar from Q for convenience, as it is clear from the context whether the polynomial or the symmetric function is meant.

then (3.10) can be written in the more compact form

$$\tau(u)\tau(v) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}z} \left(\sum_{\nu\in DP} \frac{1}{b_\nu} Q_\nu(u/v) Q_\nu(z) \right) \left(\sum_{\xi\in DP} \frac{1}{b_\xi} D(Q_\xi(u/v)) Q_\xi(\tfrac{1}{z}) \right) \tau(u)\tau(v)\,.$$

The tau function can be written, in all generality, as

$$\tau(u) = \sum_{\alpha\in DP} a^\alpha Q_\alpha(u). \tag{3.12}$$

Using arguments similar to those used for the KP hierarchy, we obtain from (3.10) the condition

$$a^\alpha a^\beta = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}z} \sum_{\eta,\xi} (-1)^{|\eta|+|\beta|} a^\eta a^\xi Q_{\bar\eta;\alpha}(z) Q_{\bar\xi;\beta}(z) \tag{3.13}$$

to be satisfied for all $\alpha, \beta \in DP$ if (3.12) is to solve the BKP hierarchy. Here we have *defined*, by analogy, composite Q-functions to be

$$Q_{\bar\nu;\mu}(z) = \sum_{\xi\in DP} (-1)^{|\xi|} \frac{b_\xi}{b_\nu} Q_{\mu/\xi}(z) Q_{\nu/\xi}(\tfrac{1}{z})\,.$$

Q-functions of a single variable are simple: $Q_\lambda(z)$ is non-zero only if λ is a one-part partition, in which case $Q_{(m)}(z) = 2z^m$ if $m \neq 0$ and $Q_{(0)}(z) = 1$. Similarly the skew Q-function $Q_{\lambda/\mu}(z)$ is non-zero only if the skew diagram $\lambda - \mu$ is a horizontal strip, in which case it is given by [19]

$$Q_{\lambda/\mu}(z) = 2^{l(\lambda)-l(\mu)} F^\lambda_{\mu\,(r)} z^r \qquad r = |\lambda| - |\mu|\,.$$

The coefficients $F^\lambda_{\mu\,(r)}$ can be calculated thus: if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ where $\lambda_i, \mu_i > 0$ for $i \leqslant n$ then

$$F^\lambda_{\mu\,(r)} = 2^{p-q}$$

where p is the number of rows for which $\lambda_j > \mu_j$, the $(n+1)$-th row excluded, and q is the number of rows for which $\lambda_{j+1} = \lambda_j$. This allows explicit calculation of the composite Q-functions, resulting in the Plücker equations

$$\begin{aligned}
&a^{(0)}a^{(3,2,1)} + a^{(2)}a^{(3,1)} - a^{(2,1)}a^{(3)} - a^{(3,2)}a^{(1)} = 0\\
&a^{(0)}a^{(4,2,1)} + a^{(2)}a^{(4,1)} - a^{(2,1)}a^{(4)} - a^{(4,2)}a^{(1)} = 0\\
&a^{(0)}a^{(4,3,1)} + a^{(3)}a^{(4,1)} - a^{(3,1)}a^{(4)} - a^{(4,3)}a^{(1)} = 0\\
&a^{(0)}a^{(5,2,1)} + a^{(2)}a^{(5,1)} - a^{(2,1)}a^{(5)} - a^{(5,2)}a^{(1)} = 0\\
&a^{(0)}a^{(5,3,1)} + a^{(3)}a^{(5,1)} - a^{(3,1)}a^{(5)} - a^{(5,3)}a^{(1)} = 0\\
&a^{(0)}a^{(6,2,1)} + a^{(2)}a^{(6,1)} - a^{(2,1)}a^{(6)} - a^{(6,2)}a^{(1)} = 0\\
&\qquad\qquad\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned} \tag{3.14}$$

The equations listed in (3.14) constitute all the non-trivial equations for $|\alpha| + |\beta| \leqslant 8$. The lack of terms $(a^\lambda)^2$ in the Plücker equations is due to the fact [6, 7] that Schur Q-polynomials solve the BKP hierarchy, i.e. given λ, $a^\alpha = \delta_{\alpha\lambda}$ is a solution to (3.13). We expect that this can be proved along the lines of appendix A—a (as yet unknown) determinantal or Pfaffian expression for composite Schur Q-functions being the key ingredient.

As with the KP hierarchy in subsection 2.3, we can obtain bilinear PDEs in $\tau(x)$ corresponding to these Plücker relations. Consider again a tau function of compound argument, which can be expanded as

$$\tau(u,w)=\sum_{\alpha}a^{\alpha}Q_{\alpha}(u,w)=\sum_{\alpha,\beta}a^{\alpha}Q_{\alpha/\beta}(u)Q_{\beta}(w)\,.$$

By the same argument as before, $a^{\beta}(u)\equiv\sum_{\beta}a^{\alpha}Q_{\alpha/\beta}(u)$ satisfies (3.13). We rewrite $a^{\beta}(u)$ by looking at its generating function in the following way:

$$\begin{aligned}\sum_{\beta}a^{\beta}(u)Q_{\beta}(t)&=\sum_{\beta}b_{\beta}^{-1}D\left(Q_{\beta}(u)\right)\tau(u)Q_{\beta}(t)\\&=D\left(\exp\left(2\sum_{m\ \mathrm{odd}}\tfrac{1}{m}p_m(t)p_m(u)\right)\right)\tau(u)\\&=\exp\left(\sum_{m\ \mathrm{odd}}p_m(t)\frac{\partial}{\partial p_m(u)}\right)\tau(u)\,.\end{aligned}$$

Now let $\partial/\partial p_m(u)=\frac{2}{m}p_m(y)$ for a fictitious set of variables $y=(y_1,y_2,\ldots)$. Then we obtain

$$a^{\beta}(u)=b_{\beta}^{-1}Q_{\beta}(y)\tau(u) \tag{3.15}$$

after using (3.5) and projecting out $Q_{\beta}(t)$. In terms of Schur Q-polynomials (3.15) becomes $a^{\beta}(u)=b_{\beta}^{-1}Q_{\beta}(\tilde{\partial}_p)\tau(u)$ where $\tilde{\partial}_p\equiv(\frac{1}{1}\partial/\partial p_1(u),(\frac{1}{2}\partial/\partial p_2(u),\ldots)$. Finally, in terms of the original BKP variables x, we have

$$a^{\alpha}(u)=b_{\alpha}^{-1}Q_{\alpha}(2\tilde{\partial})\tau(x) \tag{3.16}$$

where $\tilde{\partial}\equiv(\frac{1}{1}\partial/\partial x_1,(\frac{1}{2}\partial/\partial x_2,\ldots)$. Corresponding to the first equation of (3.14) we have the bilinear PDE

$$\begin{aligned}&\left(Q_{(0)}(2\tilde{\partial})\tau(x)\right)\left(Q_{(3,2,1)}(2\tilde{\partial})\tau(x)\right)+\left(Q_{(2)}(2\tilde{\partial})\tau(x)\right)\left(Q_{(3,1)}(2\tilde{\partial})\tau(x)\right)\\&\qquad-\left(Q_{(2,1)}(2\tilde{\partial})\tau(x)\right)\left(Q_{(3)}(2\tilde{\partial})\tau(x)\right)-\left(Q_{(3,2)}(2\tilde{\partial})\tau(x)\right)\left(Q_{(1)}(2\tilde{\partial})\tau(x)\right)=0\,.\end{aligned}$$

Using the explicit form for Schur Q-polynomials written down earlier, this can be shown to be equivalent to the Hirota bilinear form of the BKP equation

$$\left(D_1^6-5D_1^3D_3-5D_3^2+9D_1D_5\right)\tau\cdot\tau=0\,. \tag{3.17}$$

The next equation of (3.14) is a total-x_1 derivative of the BKP equation, consistent with the fact that only even degree equations are non-trivial in the BKP hierarchy [4]. The difference between the third and fourth equations of (3.14) is equivalent to the second equation of the BKP hierarchy

$$\left(D_1^8+7D_1^5D_3-35D_1^2D_3^2-21D_1^3D_5-42D_3D_5+90D_1D_7\right)\tau\cdot\tau=0$$

whereas the fifth and sixth combine to produce total derivatives of the first two BKP Hirota bilinear equations. Once again, all the bilinear equations corresponding to the Plücker equations (3.13) are expected to be equivalent to the BKP Hirota equations.

3.3. Soliton and other solutions

The vertex operator [6]

$$\Gamma(a,b)=\exp\left(\sum_{k\ \mathrm{odd}}x_k(a^k+b^k)\right)\exp\left(-2\sum_{k\ \mathrm{odd}}\frac{(a^{-k}+b^{-k})}{k}\frac{\partial}{\partial x_k}\right)$$

defines the vertex representation of b_∞ on $\mathbb{C}[x_1,x_3,\ldots]$ which underlies the BKP hierarchy. Elements of $O(\infty)$ are obtained by exponentiating elements of b_∞, and they generate new tau function solutions of the BKP hierarchy from old ones. Since $\Gamma(a,b)^2$ is zero, we are interested in operators of the type

$$\prod_i \exp\left(c_i\Gamma(a_i,b_i)\right)=\prod_i\left(1+c_i\Gamma(a_i,b_i)\right)$$

for arbitrary scalars a_i, b_i, c_i.

Via the transformations (3.9), and using manipulations analogous to the ones for the KP hierarchy, we obtain

$$\Gamma(a,b)b_\alpha^{-1}Q_\alpha(u)=(-1)^{|\alpha|}\sum_\gamma b_\gamma^{-1}Q_{\tilde{\alpha};\gamma}(a,b)Q_\gamma(u)\,. \tag{3.18}$$

More generally, we have for any n

$$\prod_{i=1}^{n}\Gamma(a_i,b_i)b_\alpha^{-1}Q_\alpha(u) \tag{3.19}$$

$$=(-1)^\alpha\prod_{1\leqslant i<j\leqslant n}\mathcal{I}\left(a_i,b_i;\tfrac{1}{a_i},\tfrac{1}{b_i}\right)\sum_\gamma b_\gamma^{-1}Q_{\tilde{\alpha};\gamma}(a_1,\ldots,a_n,b_1,\ldots,b_n)Q_\gamma(u) \tag{3.20}$$

where

$$\mathcal{I}(x,y;z,w)=\frac{(1-xz)}{(1+xz)}\frac{(1-xw)}{(1+xw)}\frac{(1-yz)}{(1+yz)}\frac{(1-yw)}{(1+yw)}\,.$$

Hence the result of the action of a product of exponentials of $\Gamma(a,b)$ on a Schur Q-function is

$$\prod_{i=1}^{N}\left(1+a_i\Gamma(a_i,b_i)\right)b_\alpha^{-1}Q_\alpha(u)$$

$$=(-1)^{|\alpha|}\sum_{n=0}^{N}\sum_{1\leqslant i_1<\cdots<i_n\leqslant N}a_{i_1}\cdots a_{i_n}\prod_{1\leqslant\eta<\zeta\leqslant n}\mathcal{I}\left(a_{i_\eta},b_{i_\eta};\frac{1}{a_{i_\zeta}},\frac{1}{b_{i_\zeta}}\right)$$

$$\times\sum_\gamma b_\gamma^{-1}Q_{\tilde{\alpha};\gamma}(a_{i_1},\ldots,a_{i_n},b_{i_1},\ldots,b_{i_n})Q_\gamma(u) \tag{3.21}$$

which are solutions of the BKP hierarchy (once the transformations (3.9) are reversed) by virtue of the fact that Schur Q-functions are also solutions. As in the KP case in subsection 2.4, when α is the zero partition we recover the known N-soliton solution [6].

4. Conclusion

In this paper, we have studied the KP and BKP hierarchies through their connections with symmetric functions. In the process we have re-derived several well known results - e.g. the fact that Schur polynomials solve the KP hierarchy, the form of the N-soliton solutions, and the relationship between Hirota derivatives and supersymmetric polynomials. We have also showed how to obtain the Plücker equations directly from the bilinear identities, a result we believe to be new. In a separate paper [13] it will be shown that symmetric function techniques also allow the derivation of closed-form expressions for Hirota polynomials of various KP-type hierarchies. There are probably other interesting properties of KP-type hierarchies which deserve to be studied from our point of view. To conclude, we note that our discussions of the KP and BKP hierarchies in sections 2 and 3 completely parallel each other. Since the S- and Schur Q-functions are but special cases ($t = 0$ and $t = -1$, respectively) of the Hall–Littlewood symmetric functions [14], other special cases (e.g. t root of unity) might be relevant to other KP-type hierarchies.

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Appendix A. Plücker equations from composite S-functions

As was shown in section 2 above, for fixed partitions α', β the presence of a term $a^{\xi}a^{\eta}$ in the associated Plücker equation necessitates non-vanishing composite S-functions $s_{\bar{\alpha}';\xi}(z)$ and $s_{\bar{\beta};\eta'}(z)$ of a single indeterminate z, such that

$$|\alpha| + |\beta| = |\xi| + |\eta| - 1\,. \tag{A.1}$$

For a single z these S-functions may be interpreted as characters of representations of $gl(1)$. By homogeneity

$$s_{\bar{\lambda};\mu}(z) = z^{|\mu|-|\lambda|} d_{\bar{\lambda};\mu} \tag{A.2}$$

where $d_{\bar{\lambda};\mu}$ is the dimension ($= 1$ for standard partitions, and 0 or ± 1 for partitions which modify to standard ones). Henceforth we consider only the dimension $d_{\bar{\lambda};\mu}$ and use the $m \times m$ determinantal expansion [20]:

$$s_{\overline{\lambda};\mu} = \det\left(s_{\overline{1^{\lambda'_i - i + m}};\mu_j - j + m} \right) \tag{A.3}$$

where $m = \max\{\lambda_1, \lambda'_1, \mu_1, \mu'_1\}$. For $gl(1)$, it can be easily established from the definition (2.13) of composite S-functions that

$$d_{\overline{1^a};b} = \begin{cases} 1 & \text{if } a = 0 \\ (-1)^b & \text{if } a = b - 1 > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{A.4}$$

It then follows from (A.4) that, since the sets $\{\lambda'_i - i + m\}$, $\{\mu_j - j + m\}$ are strictly decreasing and non-negative, precisely one non-zero ($= \pm 1$) entry per row must occur for any non-vanishing determinant; and, moreover, the non-zero entry for each successive row necessarily occurs at least one column to the right of the preceding one. Thus, except for the possibility of the last row being completely filled ($\lambda'_m = 0$), the matrix must have its non-zero entries either on the leading or (upper) next to-to-leading diagonal. In practice this means that the columns of λ are formed by either (I) adding 1 to the first $(r-1)$ rows of μ, deleting μ_r, and copying the remaining rows of μ, for $1 \leqslant r \leqslant \mu'_1$; or (II) adding one to each row of μ, together with $k (\geqslant 0)$ additional columns of length 1. The relationship between the weights of λ and μ for these cases is thus

$$\begin{array}{llll} \text{(I)} & |\lambda| = |\mu| + (r-1) - \mu_r & 1 \leqslant r \leqslant \mu'_1 & \\ \text{(II)} & |\lambda| = |\mu| + \mu'_1 + k & k \geqslant 0\,. & \end{array} \tag{A.5}$$

For the Plücker relations arising from fixed α and β the conditions (A.5) can be applied by interpreting the coefficient of $a^\xi a^\eta$ as a non-vanishing product $s_{\tilde{\xi};\alpha'} s_{\tilde{\eta}';\beta}$ such that

$$|\xi| + |\eta| = |\alpha| + |\beta| + 1. \tag{A.6}$$

In contrast to the Sato form [4], such equations can have arbitrarily many terms, as consideration of cases like $\alpha' = (s^r)$, $\beta = (q^r)$ with $r > s, p > q$ will show. For example (A.6) with (A.5) has eight solutions, and hence the corresponding Plücker relation eight terms, for the choice $\alpha' = (2^6), \beta = (6^{10})$

$$\begin{aligned} &- a^{(7^9)} a^{(2^4)} + a^{(7^8 6)} a^{(32^3)} - a^{(7^7 6^2)} a^{(3^2 2^2)} + a^{(7^6 6^3)} a^{(3^3 2)} \\ &\qquad - a^{(7^5 6^4)} a^{(3^4)} + a^{(7^2 6^7)} a^{(3^5)} - a^{(76^8)} a^{(3^5 1)} + a^{(6^9)} a^{(3^5 1^2)} = 0\,. \end{aligned}$$

In order to show that $a^\xi = \delta_{\xi\,\lambda}$ is a solution for fixed λ, it is only necessary to note that the coefficient of $(a^\lambda)^2$ is always zero, whatever α' and β. Using (A.5), $s_{\tilde{\alpha}';\lambda} s_{\tilde{\beta};\lambda'}$ is non-zero when one of the following applies:

$$\begin{array}{llll} (\mathrm{I}\alpha) & |\alpha| = |\lambda| - 1 - a_i & a_i = \lambda_i - i & 1 \leqslant i \leqslant \lambda'_1 \\ (\mathrm{II}\alpha) & |\alpha| = |\lambda| + \lambda'_1 + k & k \geqslant 0 & \\ (\mathrm{I}\beta) & |\beta| = |\lambda| - 1 - b_j & b_j = \lambda'_j - j & 1 \leqslant j \leqslant \lambda_1 \\ (\mathrm{II}\beta) & |\beta| = |\lambda| + \lambda_1 + l & l \geqslant 0 & \end{array}$$

so that $|\alpha| + |\beta|$ is one of

$$\begin{array}{ll} (\mathrm{I}\alpha) + (\mathrm{I}\beta) & (2|\lambda| - 1) - (a_i + b_j + 1) \\ (\mathrm{I}\alpha) + (\mathrm{II}\beta) & (2|\lambda| - 1) + (\lambda_1 - a_i) + l \\ (\mathrm{II}\alpha) + (\mathrm{I}\beta) & (2|\lambda| - 1) + (\lambda'_1 - b_j) + k \\ (\mathrm{II}\alpha) - (\mathrm{II}\beta) & 2|\lambda| + \lambda_1 + \lambda'_1 + k + l\,. \end{array} \tag{A.7}$$

In the first case the quantity $a_i + b_j + 1$ is never zero by Frobenius' lemma [20] which states that the sets $\{a_i\}$ and $\{b_j\}$ are disjoint (in fact, $h_{ij} > 0$ is the hook length of the node (i, j) if it belongs in λ; otherwise $-h_{ij}$ is the hook length of the node $(\lambda'_1 - i + 1, \lambda_1 - j + 1)$ of the complement λ^c of λ in its $\lambda_1 \times \lambda'_1$ rectangle). Thus $|\alpha| + |\beta|$ can never attain the required $2|\lambda| - 1$. Case four is obviously ruled out, and the second and third cases are also greater than the required $2|\lambda| - 1$ by the positive amounts $(\lambda_1 - \lambda_i) + i + l$, $(\lambda'_1 - \lambda'_j + k)$, respectively.

Appendix B. Hirota derivatives and supersymmetric Schur S- and Q-polynomials

In this appendix we derive the result

$$D_\lambda(\tau\cdot\tau)=\sum_\rho \chi_\lambda^\rho\; S_\rho\left(\tilde\partial_x/\tilde\partial_y\right)\tau(x)\tau(y)\Big|_{y=x} \tag{B.1}$$

where $D_\lambda = D_{\lambda_1}D_{\lambda_2}\cdots D_{\lambda_n}$ for any partition $\lambda=(\lambda_1,\ldots,\lambda_n)$. This result was first obtained in [21, 22] in a different guise based on the observation that a tau function solution for the KP hierarchy can be written as a Wronskian. Our derivation does not require this assumption and is thus not tied down to the KP hierarchy, and also can be readily generalized to cases where the Hirota equations involve two tau functions, as for the modified KP hierarchies. Furthermore an analogous result is available for BKP-type hierarchies.

To prove (B.1) we note that D_λ is like a power sum $p_\lambda(r)$. Hence we can think of D_n as $p_n(r)$ for a fictitious set of indeterminates r and consider the generating function

$$\sum_\lambda z_\lambda^{-1}p_\lambda(t)D_\lambda(\tau\cdot\tau). \tag{B.2}$$

Using the relations $\sum_\lambda z_\lambda^{-1}p_\lambda(t)p_\lambda(r)=\prod_{i,j}(1-t_ir_j)^{-1}$ [14] (which is yet another expression for the product on the right-hand side) and (2.8), we can write (B.2) as $\exp\left(\sum_n \frac{1}{n}p_n(t)D_n\right)(\tau\cdot\tau)$. By the definition of Hirota derivative, this is the same as $\{\exp\left(\sum_n\frac{1}{n}p_n(t)\partial_n\right)\tau\}\{\exp\left(-\sum_n\frac{1}{n}p_n(t)\partial_n\right)\tau\}$. By the usual argument of thinking of ∂_n as $p_n(w)$ (for yet another fictitious set of indeterminates w), we can rewrite the above as

$$\sum_\alpha s_\alpha(t)\{S_\alpha(\tilde\partial)\tau\}\sum_\beta(-1)^{|\beta|}s_\beta(t)\{S_{\beta'}(\tilde\partial)\tau\}.$$

On multiplying together the S-functions of t, and using the Frobenius relation (2.21) the generating function (B.2) is finally rewritten as

$$\sum_{\lambda,\rho} z_\lambda^{-1}\chi_\lambda^\rho p_\lambda(t)\sum_\beta(-1)^{|\beta|}\{S_{\rho/\beta}(\tilde\partial)\tau\}\{S_{\beta'}(\tilde\partial)\tau\}.$$

The result (B.1) follows on comparing coefficients of $p_\lambda(t)$ on both sides and using the definition (2.12) of supersymmetric S-functions. Actually, by the Frobenius relation (2.21) we have the following result

$$S_\lambda(\tilde D)(\tau\cdot\tau)=S_\lambda(\tilde\partial_x/\tilde\partial_y)\tau(x)\tau(y)\Big|_{y=x}$$

as well, which 'explains' why Hirota derivatives are intimately related to supersymmetric Schur polynomials.

The analogous expression of (B.1) for the BKP hierarchy is easily obtained. Here we are interested in D_λ for $\lambda\in OP$, the set of partitions into odd positive integers. The identity $\sum_{\lambda\in OP} z_\lambda^{-1}2^{l(\lambda)}p_\lambda(t)p_\lambda(r)=\exp\left(\sum_{n\text{ odd}}\frac{2}{n}p_n(t)p_n(r)\right)$ [14] is used to write the generating function

$$\sum_{\lambda\in OP} z_\lambda^{-1}2^{l(\lambda)}p_\lambda(t)D_\lambda(\tau\cdot\tau) \tag{B.3}$$

as $\exp\left(\sum_{n\text{ odd}}\frac{2}{n}p_n(t)D_n\right)(\tau\cdot\tau)$ which, by the definition of Hirota derivative and the identity (3.5), becomes

$$\sum_{\mu\in DP}\frac{1}{b_\mu}Q_\mu(t)\sum_{\rho\in DP}(-1)^{|\rho|}\{Q_{\mu/\rho}(2\tilde\partial)\}\{Q_\rho(2\tilde\partial)\}\,.$$

On using the relation [14]

$$Q_\mu(t) = \sum_{\lambda \in OP} z_\lambda^{-1} 2^{l(\lambda)} X_\lambda^\mu(-1) p_\lambda(u)$$

between Schur Q-functions and power sums and the definition (3.11) of supersymmetric Schur Q-functions, we finally obtain the result

$$D_\lambda(\tau \cdot \tau) = \sum_{\mu \in DP} X_\rho^\mu(-1) Q_\mu(2\tilde{\partial}_x/2\tilde{\partial}_y)\tau(x)\tau(y)\Big|_{y=x} . \tag{B.4}$$

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